Normal approximation for stochastic differential delay equations with small noises

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Content

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1. Introduction
2. Main results
3. Proofs of main results

1. Introduction

We consider the stochastic differential delay equation with small noise of the form

$$\begin{cases} X_{\varepsilon,t} = \varphi(0) + \int_0^t b(X_{\varepsilon,s}, X_{\varepsilon,s-\tau})ds + \varepsilon \int_0^t \sigma(X_{\varepsilon,s}, X_{\varepsilon,s-\tau})dB_s, \ \ t \in [0,T] \\ X_{\varepsilon,t} = \varphi(t), \ \ t \in [-\tau, 0], \end{cases} \tag{1}$$

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where the initial data $\varphi : [-\tau, 0] \to \mathbb{R}$ is a bounded deterministic function, $(B_t)_{t\in[0,T]}$ is a standard Brownian motion and b, σ are deterministic functions on $\mathbb{R}^2$.

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where the initial data $\varphi : [-\tau, 0] \to \mathbb{R}$ is a bounded deterministic function, $(B_t)_{t\in[0,T]}$ is a standard Brownian motion and b, σ are deterministic functions on $\mathbb{R}^2$.

Intuitively, as ε tends to 0, $(X_{\varepsilon,t})_{t\geq 0}$ the solution to (1) tends to $(x_t)_{t\geq 0}$, which solves the following deterministic differential delay equations

$$\begin{cases} x_t = \varphi(0) + \int_0^t b(x_s, x_{s-\tau})ds, \ t \in [0, T] \\ x_t = \varphi(t), \ t \in [-\tau, 0]. \end{cases} \tag{2}$$

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It is known from [12] that $\tilde{X}_{\varepsilon,t}$ converges to Y_t in $L^p(\Omega), p \geq 2$ as $\varepsilon \to 0$, where $(Y_t)_{t\geq 0}$ is unique solution to the following linear stochastic differential equation

$$\begin{cases} Y_t = \int_0^t \left(b_1'(x_s, x_{s-\tau})Y_s + b_2'(x_s, x_{s-\tau})Y_{s-\tau}\right) ds + \int_0^t \sigma(x_s, x_{s-\tau}) dB_s, \;\; t \in [0, T] \\ Y_t = 0, \;\; t \in [-\tau, 0]. \end{cases} \tag{4}$$

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We observe that, for each $t \in [0, T]$, Y_t is a normal random variable. Thus the sequence $(\tilde{X}_{\varepsilon,t})_{\varepsilon\in(0,1)}$ satisfies the central limit theorem as $\varepsilon \to 0$. Hence, an important problem arising here is to investigate the rate of convergence via certain distances.

1. Introduction

There are three distances commonly used in the literature.
(i) The Wasserstein distance between the laws of $\tilde{X}_{\varepsilon,t}$ and Y_t :

$$d_{\mathrm{W}}(\tilde{X}_{\varepsilon,t}, Y_t) := \sup_{|g(x)-g(y)|\leq|x-y|} |Eg(\tilde{X}_{\varepsilon,t}) - Eg(Y_t)|.$$

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(ii) The Kolmogorov distance between the laws of $\tilde{X}_{\varepsilon,t}$ and Y_t :

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(iii) The total variation distance between the laws of $\tilde{X}_{\varepsilon,t}$ and Y_t :

$$\begin{aligned} d_{\mathrm{TV}}(\tilde{X}_{\varepsilon,t}, Y_t) &:= \sup_{A\in\mathcal{B}(\mathbb{R})} |P(\tilde{X}_{\varepsilon,t} \in A) - P(Y_t \in A)| \\ &= \frac{1}{2} \sup_{\|g\|_\infty\leq 1} |Eg(\tilde{X}_{\varepsilon,t}) - Eg(Y_t)|, \end{aligned}$$

where $\mathcal{B}(\mathbb{R})$ is Borel σ-algebra on $\mathbb{R}$ and $\|g\|_\infty = \sup_{x\in\mathbb{R}} |g(x)|$.

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Similar to Theorem 1 in [12], we have

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Thus, we focus on bounding the total variation distance $d_{\mathrm{TV}}(\tilde{X}_{\varepsilon,t}, Y_t)$.

2. The main results

Assumption 2.1.

$b, \sigma : \mathbb{R}^2 \to \mathbb{R}$ *are twice differentiable functions with the partial derivatives bounded by* L.

Our Assumption 2.1 is slightly stronger than the conditions required in [12].

2. The main results

Theorem 2.1.

Let Assumption 2.1 hold. Consider the stochastic processes $(\tilde{X}_{\varepsilon,t})_{-\tau\le t\le T}$ *and* $(Y_t)_{-\tau\le t\le T}$ *defined by (3) and (4), respectively. Then, we have*

$$d_{\mathrm{TV}}(\tilde{X}_{\varepsilon,t}, Y_t) \le \frac{Ct\varepsilon}{\sqrt{\mathrm{Var}(Y_t)}} \ \forall\, \varepsilon \in (0,1), 0 < t \le T, \tag{5}$$

where C *is a positive constant not depending on* t *and* ε*.*

We also show that the convergence rate is of optimal order.

2. The main results

Theorem 2.2.

Let Assumption 2.1 hold. We additionally assume that the partial derivatives of b and σ are continuous. Then, for any continuous and bounded function g, we have

$$\lim_{\varepsilon\to 0} \frac{Eg(\tilde{X}_{\varepsilon,t}) - Eg(Y_t)}{\varepsilon} = \frac{1}{2\mathrm{Var}(Y_t)} E\left[g(Y_t)\delta\left(Z_t DY_t\right)\right], \ 0 < t \leq T, \tag{6}$$

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where $Z_t=0$ *for* $t\in[-\tau,0]$ *and*

$$Z_t=\int_0^t b_1'(x_s,x_{s-\tau})Z_s ds+\int_0^t b_2'(x_s,x_{s-\tau})Z_{s-\tau}ds$$
$$+\int_0^t\left(b_{11}''(x_s,x_{s-\tau})Y_s^2+b_{12}''(x_s,x_{s-\tau})Y_sY_{s-\tau}+b_{21}''(x_s,x_{s-\tau})Y_sY_{s-\tau}+b_{22}''(x_s,x_{s-\tau})Y_{s-\tau}^2\right)ds$$
$$+2\int_0^t\left(\sigma_1'(x_s,x_{s-\tau})Y_s+\sigma_2'(x_s,x_{s-\tau})Y_{s-\tau}\right)dB_s,\ t\in[0,T]. \tag{7}$$

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In particular, we have

$$\lim_{\varepsilon \to 0} \frac{d_{\mathrm{TV}}(\tilde{X}_{\varepsilon,t}, Y_t)}{\varepsilon} \geq \frac{1}{2\mathrm{Var}(Y_t)} E\left|E\left[\delta(Z_t DY_t)|Y_t\right]\right|, \; 0 < t \leq T. \quad (8)$$

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Lemma 3.1.

Let $F_1 \in \mathbb{D}^{2,4}$ be such that $\|DF_1\|_{L^2[0,T]} > 0$ a.s. Then, for any random variable $F_2 \in \mathbb{D}^{1,2}$ and any measurable function g with $\|g\|_\infty = \sup_{x\in\mathbb{R}} |g(x)| \leq 1$, we have

$$|Eg(F_1) - Eg(F_2)| \leq C\left(E\|DF_1\|^{-8}_{L^2[0,T]}E\left(\int_0^T\int_0^T |D_\theta D_r F_1|^2 d\theta dr\right)^2 + (E\|DF_1\|^{-2}_{L^2[0,T]})^2\right)^{\frac{1}{4}}\|F_1 - F_2\|_{1,2}, \tag{9}$$

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This lemma is Theorem 3.1 in the recent paper [5].

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Proposition 3.2.

Suppose Assumption 2.1. Let $(\tilde{X}_{\varepsilon,t})_{-\tau\le t\le T}$ and $(Y_t)_{-\tau\le t\le T}$ are defined by (3) and (4). Then, for every $p \geq 2$, we have

$$E|\tilde{X}_{\varepsilon,t} - Y_t|^p \leq Ct^p\varepsilon^p \; \forall \varepsilon \in (0,1), 0 \leq t \leq T,$$

where C is a positive constant not depending on t and ε.

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Proposition 3.3.

Suppose Assumption 2.1. Let $(\tilde{X}_{\varepsilon,t})_{-\tau\leq t\leq T}$ *and* $(Y_t)_{-\tau\leq t\leq T}$ *are defined by (3) and (4). Then we have*

$$\|D\tilde{X}_{\varepsilon,t} - DY_t\|^2_{L^2[0,T]} \leq Ct^2\varepsilon^2 \ \forall\, \varepsilon \in (0,1), 0 \leq t \leq T, \tag{10}$$

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$$\|DY_t\|^2_{L^2[0,T]} = \mathrm{Var}(Y_t).$$

We apply Lemma 3.1 to get

$$|Eg(\tilde{X}_{\varepsilon,t}) - Eg(Y_t)| \leq \frac{Ct\varepsilon}{\sqrt{\mathrm{Var}(Y_t)}}.$$

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we obtain

$$\lim_{\varepsilon\to 0}\left(\frac{E[g(\tilde{X}_{\varepsilon,t})] - E[g(Y_t)]}{\varepsilon} - \frac{1}{2\mathrm{Var}(Y_t)}\left(E\left[g(Y_t)Z_tY_t\right] - E\left[g(Y_t)\langle DZ_t, DY_t\rangle\right]\right)\right) = 0. \tag{13}$$

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We consider the function $h(x) = \operatorname{sign}\left(E\left[\delta\left(Z_t DY_t\right) \middle| Y_t = x\right]\right)$ for $x \in \mathbb{R}$.

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Applying (6) and letting $n \to \infty$, we obtain (8).

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Thank You!